Numerical Methods for PDEs

Hyperbolic PDEs: Coupled system/Nonlinear conservation laws/A nonlinear Lax-Wendroff scheme

(Lecture 18, Week 6)

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Outline

Wave Equation as Coupled Advection Equations

We can decouple the wave equation

$$u_{tt} = a^2 u_{xx}$$

as a pair of coupled first order equations

$$\textbf{(CAE)} \qquad \begin{cases} u_t + a v_x = 0 \\ v_t + a u_x = 0, \end{cases}$$

which is supplemented with the physical initial conditions $u(x,0)$ and $v(x,0)$.

Check this: Differentiate the first equation with respect to t, the second with respect to x, then eliminate v_{xt}

Vector form of (CAE):

$$\mathbf{u}_t + A\mathbf{u}_x = 0, \quad \text{where} \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

Lax-Wendroff scheme **(LW)** for **(CAE)**

Basic idea: Include dissipation by a Taylor expansion in time up to 2nd order and use the relation $\left(\frac{\partial}{\partial t}\right)^m u = \left(-a\frac{\partial}{\partial x}\right)^m u$, such that

$$\begin{aligned}
\mathbf{u}(x,t+k) &= \mathbf{u} + k\mathbf{u}_t + \frac{1}{2}k^2\mathbf{u}_{tt} + O(k^3)\Big|_{(x,t)}, \\
&= \mathbf{u} - Ak\mathbf{u}_x + \frac{1}{2}A^2k^2\mathbf{u}_{xx} + O(k^3)\Big|_{(x,t)}, \\
&\approx \mathbf{u} - Ak\frac{D_x}{2h}\mathbf{u} + \frac{1}{2}A^2k^2\frac{\delta_x^2}{h^2}\mathbf{u}.
\end{aligned}$$

This suggests the following explicit scheme

$$\mathbf{w}_j^{n+1} = \mathbf{w}_j^n - \frac{1}{2}P(\mathbf{w}_{j+1}^n - \mathbf{w}_{j-1}^n) + \frac{1}{2}P^2(\mathbf{w}_{j+1}^n - 2\mathbf{w}_j^n + \mathbf{w}_{j-1}^n)$$

where $\mathbf{w} = \begin{pmatrix} w \\ z \end{pmatrix} \approx \begin{pmatrix} u \\ v \end{pmatrix}$, $P = \frac{k}{h}A$. For I the 2×2 unit matrix

(LW) $$\mathbf{w}_j^{n+1} = (I - P^2)\mathbf{w}_j^n - \frac{1}{2}P(I-P)\mathbf{w}_{j+1}^n + \frac{1}{2}P(I+P)\mathbf{w}_{j-1}^n.$$

Nonlinear conservation laws

So far: *Linear* hyperbolic problems

Many real problems: Require *nonlinear* equations, e.g. fluid and gas flow

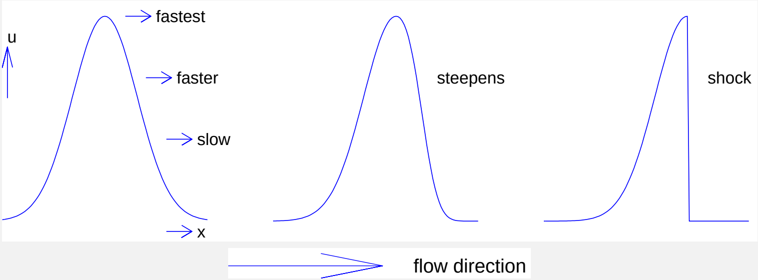
Nonlinear (AE):

$$u_t + a(u)u_x = 0$$

which includes the *Burgers' equation*

$$\textbf{(BE)} \qquad u_t + u\, u_x = 0.$$

Properties of (BE): "Top" moves faster than the "tail" on either side and eventually overtakes the tail $\Rightarrow$ formation of a shock



This figure shows a flow in positive x-direction, i.e., $a(u) > 0$.

Conservation law (CL): The numerical approximation of $u_t + a(u)u_x = 0$ leads often to unstable results. Rewriting it as follows

$$\textbf{(CL)} \qquad u_t + a(u)u_x = 0 \overset{\text{rearrange}}{\rightarrow} u_t + \frac{\partial}{\partial x}(F(u)) = 0$$

improves properties of associated numerical shemes where

$$\frac{d}{du}F(u) = a(u), \quad \text{since} \quad \frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}.$$

Example: Burgers' equation: $a(u) = u \quad \Rightarrow \quad F(u) = 1/2u^2$

Nonlinear (CL): A Lax-Wendroff scheme

Goal: Extend the Lax-Wendroff method to nonlinear conservation laws

$$\textbf{(NLCL)} \qquad u_t + [F(u)]_x = 0.$$

Idea: Apply the same strategy as for the advection equation:
Step 1: Use a truncated Taylor series for $u(x, t + k)$, i.e.,

$$u(x, t+k) \approx \left[u + k\, u_t + \frac{1}{2} k^2 u_{tt} \right]_{(x,t)} .$$

Step 2: Replace the time derivatives by space derivatives, that is,

$$\begin{aligned} u_t &= -[F(u)]_x \\ u_{tt} &= \frac{\partial}{\partial t}\left(-[F(u)]_x\right) = -\frac{\partial}{\partial x}\left([F(u)]_t\right) = -\frac{\partial}{\partial x}\left(F'(u) \cdot u_t\right) \\ &= -\frac{\partial}{\partial x}\left(F'(u)[-F(u)]_x\right) = \frac{\partial}{\partial x}\left(F'(u)\frac{\partial}{\partial x}F(u)\right) . \end{aligned}$$

Step 3: Defining

$$Q(u) = F'(u)\frac{\partial}{\partial x}F(u), \text{ so that } u_{tt} = \frac{\partial}{\partial x}Q(u).$$

leads to

$$[*] \qquad u(x_j, t_{n+1}) \approx \left[u - k\,\frac{\partial}{\partial x}F(u) + \frac{1}{2}k^2\frac{\partial}{\partial x}Q(u)\right]_{(x_j,t_n)}.$$

Step 4: Approximate x-derivatives by central differences:

$$\frac{\partial}{\partial x}F(u)\Big|_{(x_j,t_n)} \approx \frac{F(w_{j+1}^n) - F(w_{j-1}^n)}{2h} = \frac{F_{j+1}^n - F_{j-1}^n}{2h},$$

$$\frac{\partial}{\partial x}Q(u)\Big|_{(x_j,t_n)} \approx \frac{Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n}{h},$$

where we applied a spacing of $h/2$ w.r.t. Q.

We have also used the notation

$$Q^n_{j+\frac{1}{2}} = F'(w^n_{j+\frac{1}{2}}) \frac{\partial}{\partial x} F(u)\Big|_{(x_{j+\frac{1}{2}}, t_n)} \approx F'(w^n_{j+\frac{1}{2}}) \left(\frac{F^n_{j+1} - F^n_j}{h} \right)$$

so

$$\frac{\partial}{\partial x} Q(u)\Big|_{(x_j, t_n)} \approx \frac{1}{h^2} \left\{ F'(w^n_{j+\frac{1}{2}}) \left(F^n_{j+1} - F^n_j \right) - F'(w^n_{j-\frac{1}{2}}) \left(F^n_j - F^n_{j-1} \right) \right\}.$$

Since we do not have values for u at half-integer points $x_{j+\frac{1}{2}}$, we further approximate

$$F'(w^n_{j+\frac{1}{2}}) \approx \frac{1}{2} \left(F'(w^n_j) + F'(w^n_{j+1}) \right)$$

and similarly for $F'(w^n_{j-\frac{1}{2}})$.

Step 5: Inserting these findings into **[∗]** gives

$$\textbf{(NLW)} \quad \begin{cases} w_j^{n+1} = w_j^n - \frac{k}{2h}\left(F_{j+1}^n - F_{j-1}^n\right) + \frac{k^2}{2h^2}\Big\{a_{j+\frac{1}{2}}^n\left[F_{j+1}^n - F_j^n\right] \\ \qquad -a_{j-\frac{1}{2}}^n\left[F_j^n - F_{j-1}^n\right]\Big\} \end{cases}$$

where $\qquad a_{j+\frac{1}{2}}^n = \frac{1}{2}\left[F'(w_j^n) + F'(w_{j+1}^n)\right] \approx F'(w_{j+\frac{1}{2}}^n).$

Remark: The **(NLW)** scheme reduces to the linear **(LW)** for $F(u) = au$, $a = \text{const}$, i.e. when $F' = a$.

Check:

$$\begin{aligned} w_j^{n+1} &= w_j^n - \frac{k}{2h}\left(aw_{j+1}^n - aw_{j-1}^n\right) + \frac{k^2}{2h^2}\Big\{a\left[aw_{j+1}^n - aw_j^n\right] \\ &\quad - a\left[aw_j^n - aw_{j-1}^n\right]\Big\} \\ &= w_j^n - \frac{p}{2}\left(w_{j+1}^n - w_{j-1}^n\right) + \frac{p^2}{2}\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right), \quad p = \frac{ak}{h} \end{aligned}$$

which is the formula derived earlier.

Stability of **(NLW)**

In the linear scheme we have stability only for $|p| \leq 1$. In the nonlinear scheme, $F'(u)$ plays the role of a, so we should require that

$$\left|F'(w_j^n)\right| \frac{k}{h} \leq 1$$

for all m. For safety we replace "≤ 1" by "$= 0.9$" and require

$$k = \frac{0.9h}{\max_j \left|F'(w_j^n)\right|}$$

Note that k will vary from step to step.