Numerical Methods for PDEs

Elliptic PDEs: Central difference approximation/Local truncation error/Convergence

(Lecture 19, Week 7)

Markus Schmuck

Department of Mathematics and Maxwell Institute for Mathematical Sciences
Heriot-Watt University, Edinburgh

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Outline

1. **Poisson equation:** Central difference approximation
2. **Poisson equation:** Local truncation error
3. **Poisson equation:** Convergence

Poisson equation: Classical solution and examples

Let $\Omega \subset \mathbb{R}^d$ be open and bounded with boundary $\partial\Omega$ (i.e., $\overline{\Omega} = \Omega \cup \partial\Omega$) and $\hat{\mathrm{A}} = \{a_{ij}\}_{1 \leq i,j \leq d}$ a symmetric, positive definite matrix.

Classical solutions: *A function $u \in C^2(\Omega)$ that solves*

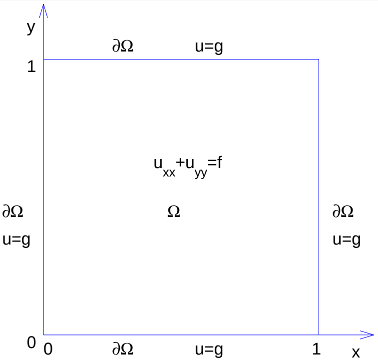
$$\textbf{(PE)} \quad \begin{cases} -\mathrm{div}\left(\hat{\mathrm{A}} \nabla u\right) = f & \text{in } \Omega \\ u(x) = g(x) & \text{on } \partial\Omega \end{cases}$$

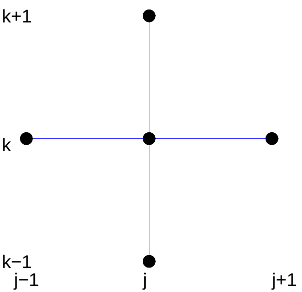
is called **classical solution** *of the Poisson Equation* **(PE)**.

Applications:

- Stationary distribution of heat
- Stationary fluid flow in homogeneous & porous media
- Stationary electric potential

For a uniform mesh with $h = h_x := 1/J = h_y := 1/K$, denote the nodal interpolation by $f_{j,k} = f(x_j, y_k)$ and the numerical approximation by $w_{j,k} \approx u(x_j, y_k)$ for $j = 0, 1, 2, \ldots, J$ and $k = 0, 1, 2, \ldots, K$.

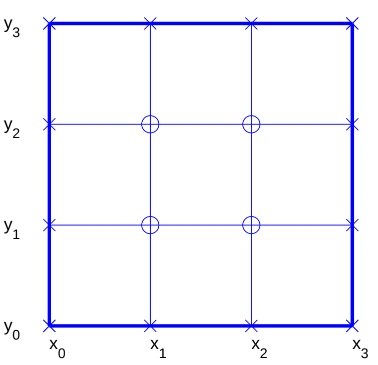




Central difference approximation in 2D:

(CDA) $$w_{j-1,k} + w_{j+1,k} + w_{j,k-1} + w_{j,k+1} - 4w_{j,k} = h^2 f_{j,k}.$$

(CDA) as a coupled system: For $J = K = 3$ we get



$$
\begin{array}{l|l}
w_{0,1} + w_{2,1} + w_{1,0} + w_{1,2} - 4w_{1,1} = \frac{1}{9} f_{1,1} & w_{2,1} + w_{1,2} - 4w_{1,1} = \frac{1}{9} f_{1,1} - g_{0,1} - g_{1,0} \\
w_{0,2} + w_{2,2} + w_{1,1} + w_{1,3} - 4w_{1,2} = \frac{1}{9} f_{1,2} & w_{2,2} + w_{1,1} - 4w_{1,2} = \frac{1}{9} f_{1,2} - g_{0,2} - g_{1,3} \\
w_{1,1} + w_{3,1} + w_{2,0} + w_{2,2} - 4w_{2,1} = \frac{1}{9} f_{2,1} & w_{1,1} + w_{2,2} - 4w_{2,1} = \frac{1}{9} f_{2,1} - g_{3,1} - g_{2,0} \\
w_{1,2} + w_{3,2} + w_{2,1} + w_{2,3} - 4w_{2,2} = \frac{1}{9} f_{2,2} & w_{1,2} + w_{2,1} - 4w_{2,2} = \frac{1}{9} f_{2,2} - g_{3,2} - g_{2,3}
\end{array}
$$

where we took the boundary condition g into account in the system on the right-hand side.

Matrix form of (CDA): These coupled equations can be written as

$$S\mathbf{w} = \mathbf{b}$$

where

$$\mathbf{w} = \begin{pmatrix} w_{1,1} \\ w_{1,2} \\ \vdots \\ w_{1,K-1} \\ w_{2,1} \\ \vdots \\ w_{J-1,K-1} \end{pmatrix}$$

and

$$S = \begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} \frac{1}{9} f_{1,1} - g_{0,1} - g_{1,0} \\ \frac{1}{9} f_{1,2} - g_{0,2} - g_{1,3} \\ \frac{1}{9} f_{2,1} - g_{3,1} - g_{2,0} \\ \frac{1}{9} f_{2,2} - g_{3,2} - g_{2,3} \end{pmatrix}.$$

Numerical example:

Take $f(x,y) = 1$ and $g(x,0) = \sin\pi x, g(x,1) = g(0,y) = g(1,y) = 0$. In this case the rhs vector is

$$\mathbf{b} = \begin{pmatrix} \frac{1}{9} - \frac{\sqrt{3}}{2} \\ \frac{1}{9} - \frac{\sqrt{3}}{2} \\ \frac{1}{9} \\ \frac{1}{9} \end{pmatrix},$$

since $\sin(\pi/3) = \sqrt{3}/2$. Solving this system by Gaussian elimination we get

$$\mathbf{w} = \begin{pmatrix} w_{1,1} \\ w_{1,2} \\ w_{2,1} \\ w_{2,2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{18} + \frac{3\sqrt{3}}{16} \\ -\frac{1}{18} + \frac{3\sqrt{3}}{16} \\ -\frac{1}{18} + \frac{\sqrt{3}}{16} \\ -\frac{1}{18} + \frac{\sqrt{3}}{16} \end{pmatrix} = \begin{pmatrix} 0.26920 \\ 0.26920 \\ 0.05270 \\ 0.05270 \end{pmatrix}$$

Note the symmetry $w_{1,1} = w_{1,2}, \quad w_{2,1} = w_{2,2}$, which we would expect since the BCs are symmetric about the line $x = 1/2$.

Problem: For small mesh parameters $h = \Delta x = \Delta y$, the matrix S will be sparse and large.

Solution: Apply iterative methods such as Gauss-Seidel or SOR.

Poisson equation: Local truncation error

For

$$L_\Delta w_{j,k} = \frac{w_{j-1,k} - 2w_{j,k} + w_{j+1,k}}{h^2} + \frac{w_{j,k-1} - 2w_{j,k} + w_{j,k+1}}{h^2},$$

the numerical solution satisfies $L_\Delta w_{j,k} - f_{j,k} = 0$. Then, the LTE reads

$$\begin{aligned}
\text{LTE} &= L_\Delta u(x_j, y_k) - f_{j,k} = \frac{u(x_j - h, y_k) - 2u(x_j, y_k) + u(x_j + h, k)}{h^2} + \\
&\quad + \frac{u(x_j, y_k - h) - 2u(x_j, y_k) + u(x_j, y_k + h)}{h^2} - f_{j,k} \\
&= \left[u_{xx} + \frac{1}{12} h^2 u_{xxxx} + u_{yy} + \frac{1}{12} h^2 u_{yyyy} - f_{j,k} + O(h^4) \right]_{(x_j, y_k)} .
\end{aligned}$$

Using $u_{xx} + u_{yy} = f$ finally leads to

$$\text{LTE} = \frac{1}{12} h^2 (u_{xxxx} + u_{yyyy}) + O(h^4).$$

Hence the scheme is consistent with order $m = 2$ for $u \in C^{m+2}(\Omega)$

Poisson equation: Convergence

Define the error as $z_{j,k} = u(x_j, y_k) - w_{j,k}$.

Convergence: *We say that the numerical scheme is convergent if and only if* $|z_{j,k}| \to 0$ *as* $h \to 0$ *for all* j, k*.*

Lemma: (Convergence) *The numerical solution* $w_{j,k}$ *of the Central Difference Approximation* **(CDA)** *converges to the classical solution* $u \in C^4(\Omega)$ *of the Poisson Equation* **(PE)**.

Before we prove this Lemma, we establish the

Lemma: (Discrete Maximum Prinicple (DMP)) *If* $L_\Delta m_{j,k} \geq 0$ *for all* (j, k) *and any function* $m_{j,k}$*, then* $m_{j,k}$ *attains its maximum value* $M = m_{j^*,k^*}$ *on the boundary, i.e., at* $(x_{j^*}, y_{k^*}) \in \partial\Omega$*.*

Proof: Discrete Maximum Principle (DMP)

Let $M = \max_{j,k=0,\dots,J}[m_{j,k}]$, and assume this is attained at an interior point, say (j^*, k^*). This implies that

$$
\begin{aligned}
& m_{j^*-1,k^*} + m_{j^*+1,k^*} + m_{j^*,k^*-1} + m_{j^*,k^*+1} - 4M \geq 0 \\
\text{i.e. } M &\leq \frac{1}{4}(m_{j^*-1,k^*} + m_{j^*+1,k^*} + m_{j^*,k^*-1} + m_{j^*,k^*+1}) \\
&\leq \frac{1}{4}(4M) = M.
\end{aligned}
$$

Equality is only possible if all interior points and their neighbours take the value M, so the maximum value must be attained somewhere on the boundary. (We have assumed here that $M \geq 0$ which is sufficient for our needs).

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Proof: Convergence

Apply the finite difference operator L_Δ to $z_{j,k}$, that is,

$$L_\Delta z_{j,k} = L_\Delta u(x_j, y_k) - L_\Delta w_{j,k} = L_\Delta u(x_j, y_k) - f_{j,k} = \text{LTE}.$$

We now introduce a *comparison function* $C_{j,k} = x_j^2 + y_k^2$. We have

$$L_\Delta C_{j,k} = \frac{(x_j - h)^2 - 2x_j^2 + (x_j + h)^2}{h^2} + \frac{(y_k - h)^2 - 2y_k^2 + (y_k + h)^2}{h^2} = 4,$$

and $C_{j,k}$ is non-negative with a maximum value of 2 at $x = y = 1$. Let us define another function $c_{j,k}$ on the mesh by

$$c_{j,k} = z_{j,k} + \frac{1}{4} C_{j,k} |\text{LTE}|_{\max}$$

so

$$\begin{aligned} L_\Delta c_{j,k} &= \text{LTE} + \frac{1}{4} \times 4\, |\text{LTE}|_{\max} \\ &\geq 0, \quad \forall (x_j, y_k) \in \Omega \qquad (*) \end{aligned}$$

With the (DMP) and **(∗)**, $c_{j,k}$ attains its maximum on the boundary. However $z_{j,k}$ vanishes on the boundary since B.C. exact. Moreover, the maximum of $C_{j,k}$ (on boundary) is 2, so $\max(c_{j,k}) = \frac{1}{2}|\text{LTE}|_{\text{max}}$. Now we have

$$z_{j,k} = c_{j,k} - \frac{1}{4} C_{j,k} |\text{LTE}|_{\text{max}} \quad \text{by definition}$$
$$\leq c_{j,k}, \quad \forall j,k, \quad \text{since } C_{j,k} \text{ is non-negative}$$
$$\text{so } z_{j,k} \leq \frac{1}{2} |\text{LTE}|_{\text{max}}.$$

We can similarly repeat the analysis to place a lower bound on $z_{j,k}$. Define

$$\bar{c}_{j,k} = -z_{j,k} + \frac{1}{4} C_{j,k} |\text{LTE}|_{\text{max}}$$
$$\text{so } L_\Delta \bar{c}_{j,k} = -\text{LTE} + |\text{LTE}|_{\text{max}}$$
$$\geq 0, \quad \forall j,k.$$

so the max of $\bar{c}_{j,k}$ is attained on the boundary and is equal to $\frac{1}{2}|\text{LTE}|_{\text{max}}$.

Hence

$$
\begin{aligned}
z_{j,k} &= -\bar{c}_{j,k} + \frac{1}{4} C_{j,k} |\mathrm{LTE}|_{\max} \\
&\geq -\bar{c}_{j,k} \qquad \text{since } C_{j,k} \text{ is nonnegative} \\
&\geq -\frac{1}{2} |\mathrm{LTE}|_{\max}.
\end{aligned}
$$

We have finally that

$$
\boxed{-\frac{1}{2}|\mathrm{LTE}|_{\max} \leq z_{j,k} \leq \frac{1}{2}|\mathrm{LTE}|_{\max}}
$$

so $z_{j,k} \to 0$ as $h \to 0$ since the scheme is consistent. This is sufficient to prove convergence.

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