Numerical Methods for PDEs

Stability of the θ -Method and Extensions

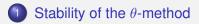
(Lecture 7, Week 3)

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Edinburgh, 26 January, 2015









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M. Schmuck (Heriot-Watt University) Numerical Methods for PDEs, Lecture 7

For $r = k/h^2$, the θ -method reads

$$\begin{aligned} &-\theta r w_{m-1}^{n+1} + (1+2\theta r) w_m^{n+1} - \theta r w_{m+1}^{n+1} \\ &= (1-\theta) r w_{m-1}^n + (1-2(1-\theta)r) w_m^n + (1-\theta) r w_{m+1}^n \,. \end{aligned}$$

Then, substitute $w_m^n = \xi^n e^{im\omega}$ and simplify

 $-\theta r e^{i(m-1)\omega} \xi^{n+1} + (1+2\theta r) e^{im\omega} \xi^{n+1} - \theta r e^{i(m+1)\omega} \xi^{n+1} = (1-\theta) r e^{i(m-1)\omega} \xi^n + (1-2(1-\theta)r) e^{im\omega} \xi^n + (1-\theta) r e^{i(m+1)\omega} \xi^n$



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$$e^{i\omega} - 2 + e^{-i\omega} = -2(1 - \cos(\omega)) = -4\sin^2(\omega/2)$$

in the above equation gives

 $\xi + 4\xi heta\sin^2(\omega/2)r = 1 - 4(1- heta)r\sin^2(\omega/2)$



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We need $|\xi| \le 1$ for stability for all $\omega \in [-\pi, \pi]$. Since ξ is clearly real in this case this means we require $-1 \le \xi \le 1$. Now

$$\xi = \frac{1 + 4\theta r \sin^2(\omega/2) - 4r \sin^2(\omega/2)}{1 + 4\theta r \sin^2(\omega/2)} = 1 - \frac{4r \sin^2(\omega/2)}{1 + 4\theta r \sin^2(\omega/2)}$$

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(i) If $\theta \ge 1/2$, this last inequality will clearly hold for *all r*. (ii) If $\theta < 1/2$, we need in the worst case ($\omega = \pi$) that

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Remark. If $\theta = 0$, we recover the familiar $r \le 1/2$ result for the FTCS scheme.



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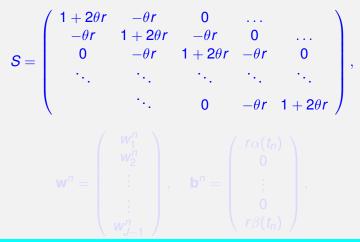


Matrix form of the θ -method

We can write the θ -method in much the same form as the BTCS scheme,

$$S\mathbf{w}^{n+1} = M\mathbf{w}^n + (1-\theta)\mathbf{b}^n + \theta\mathbf{b}^{n+1}$$

where



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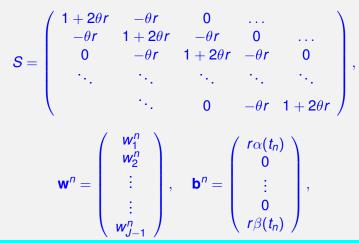
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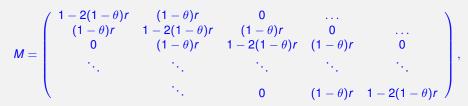
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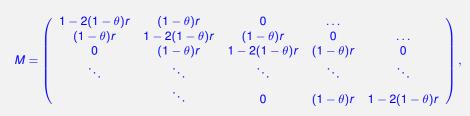


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(i) Set \mathbf{q} = M\mathbf{w}^n + (1 - \theta)\mathbf{b}^n + \theta\mathbf{b}^{n+1}
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(iii) Set w^{n+1} = v
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The matrix **S** is tridiagonal (if $\theta > 0$), so solving (ii) is fairly quick and easy. This is still more work than solving the explicit FTCS scheme ($\theta = 0$) but not much more.





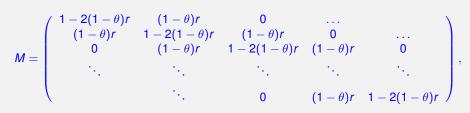
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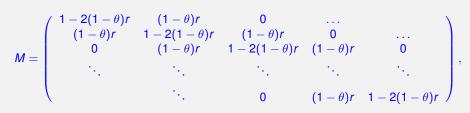


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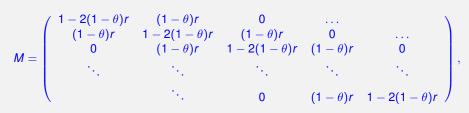
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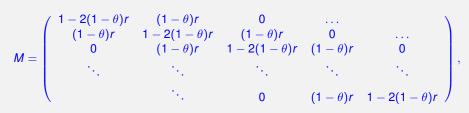


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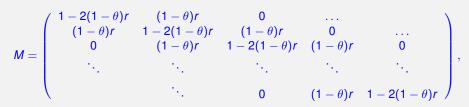


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(Note the accuracy of some schemes can be increased by choosing a special value for r).

Remark. 1. FTCS scheme easy to apply (because it is explicit), but the time step constraint for stability requires $k \le 1/2h^2$, such that for *h* small, we have *k* very small.

2. The θ -method for $\theta > 1/2$ allows a larger time step for stability (but not too large, otherwise the LTE gets big), and hence can require less overall computing.

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